Stability of Nash equilibrium in the private provision of public goods

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Abstract

The purpose of this paper is to show that the necessary and sufficient conditions of the local stability for Nash equilibrium in the private provision of public goods. Based on our proposition for the stability, we show the effectiveness of the concept of partial stability of the Nash equilibrium. Even if the Nash equilibrium of the society is instable, there exists a maximum group and the adjustment of members of the group converges to the Nash equilibrium provided that the other members in the society supply public goods at the level of Nash equilibrium.

JEL classification: H41; C72; C62
Kye words: Nash Equilibrium; Stability; Public Goods

1 Introduction

Nash equilibrium plays important roles for understanding of features of public goods. Although the equilibrium is generally not efficient, it provides a standard point for us to see rationality of players in the private provision of public goods. We are able to have some reasons why players chose the equilibrium point. For example, rational inference of players may lead their moves to the equilibrium, or if a unique outcome is brought in the game, it should be the Nash equilibrium. We need, however, more positive reasons for supporting the Nash equilibrium.

Bergstrom [2, 3] provided a proof of the uniqueness of the Nash equilibrium under the assumption that both private and public goods are normal. The assumption is so weak that under general settings we can concentrate on analyzing this unique Nash equilibrium.

One of crucial features of the equilibrium is related to stability. If the spontaneous moves based upon private motivation lead to the Nash equilibrium, we

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can naturally see that the Nash equilibrium have strong reality. Unfortunately we have few papers relating to the stability of the Nash equilibrium in the provision of public goods. Cornes [1] touched on the stability and showed that in case of two players if there are multiple points of Nash equilibrium, stable points and instable points appear one after another. If the equilibrium for a public goods game by two players is unique, the equilibrium should be stable.

In n-players games of public goods, even if the Nash equilibrium is unique, the equilibrium is not necessary stable. This may be a reason why so far we have not been able to accumulate knowledge on the stability. However, the fact does not mean that the issues related to the stability are not important. Considering that the stability of Nash equilibrium is connected to the understanding of our society, we should not resign to analyze the condition of the stability.

The issues of stability of the Nash equilibrium in the private public goods are closely related to the Cournot-Nash equilibrium on the Oligopoly problems. The stability of Cournot equilibrium had been intensively studied since 1960's. The discussion was begun by Theocharis [13] and there had appeared many papers concerning this topics, for example, Fisher [4], Hahn [6], Okuguchi [8, 9], Zhang [14], Szidarovszky [11].

Theocharis [13] showed that the iteration with two players brought stability compared with instability of more than two players. Thereafter, the direction of theoretical extension had been to eliminate the assumptions of instantaneous adjustment and to use continuous time models. Under those assumption, we could see some stable cases for the model with many players.

The purpose of this paper is to show that the necessary and sufficient conditions of the local stability for Nash equilibrium in the private provision of public goods. Players of our model instantaneously adjust the levels of the provision of public goods. On this point, we distinguish our model from the model using for discussions for stability of Cournot-Nash equilibrium. Our assumptions for instantaneous adjustment are appropriate in the situation that players of the game have strategies and explicit reaction functions. Therefore our model has similarity with original model of Theocharis [13].

Based on our proposition for the stability, we show the effectiveness of the concept of partial stability of the Nash equilibrium. Even if the Nash equilibrium of the society is instable, there exists a maximum group and the adjustment of members of the group converges to the Nash equilibrium provided that the other members in the society supply public goods at the level of Nash equilibrium. We can prove all partial groups included in the maximum group are also stable. Furthermore we show the scale dependency of the stability.

2 Assumptions and the Nash Equilibrium

Suppose that there are n players on the game of the private provision of public goods. The player i consumes x_i private goods and g_i public goods in a period. Since the change of the price level is not important, we assume that the prices of one unit of both goods are adjusted as unity. ith player has constant income

w_i in every period. Therefore the budget constraint is expressed as follows.

$$w_i = x_i + g_i \tag{1}$$

The quantity of the public goods provided by the other players than i is defined as G_{-1}. That is,

$$G_{-i} = g_1 + g_2 + \cdots + g_{i-1} + g_{i+1} + \cdots + g_n \qquad i = 1, 2, \cdots, n.$$

The utility function $u^i(x_i, g_i + G_{-i}), i = 1, 2, \cdots, n$ have basic properties as defined in consumer demand theories. We should mention, for the later discussions, that this utility function is continuously differentiable for as often as is necessary.

Let us pose some fundamental features of our model.

Assumption 1 : Both private goods and public goods are normal for every player.

In the private provision of public goods, player i maximize the utility function under (1) and given G_{-i}. The first order condition is as follows.

$$-u^i_x(w_i - g_i, g_i + G_{-i}) + u^i_g(w_i - g_i, g_i + G_{-i}) = 0 \qquad i = 1, 2, \cdots, n, \tag{2}$$

where $u^i_x = \partial u^i / \partial x_i, u^i_g = \partial u^i / \partial g_i$.The ith equation implicitly shows the reaction function of player i for given G_{-i}. As is well known, Nash equilibrium $(g^*_1, g^*_2, \cdots, g^*_n)$ satisfies all those reaction functions. This assumption ensures the uniqueness of the Nash equilibrium (See Bergstrom [2, 3]).

Assumption 2 : Every player provides positive public goods at the Nash equilibrium.

This assumption obviously restricts our discussion. However, since we focus on the local stability and depend upon the differentiability of the equilibrium, we have to exclude possible corner solutions for some players who do not provide public goods.

3 Stability Condition of the Nash Equilibrium

In the first order conditions (2) , for a given G_i in period t, the optimal value of g_i in period $t + 1$ is uniquely given ($t = 0, 1, 2, \cdots$). Therefore we have explicit reaction functions.

$$g_i = R_i(G_{-1}) \qquad i = 1, 2, \cdots, n$$

Those functions show that players completely and instantaneously adjusted so as to maximize their utility based upon the provision of other players. Since each function continuously differentiable at least in the neighborhood of the Nash equilibrium, we have the Jacobian matrix R.

$$R = \begin{pmatrix} 0 & -s_1 & \cdots & -s_1 \\ -s_2 & 0 & \cdots & -s_2 \\ \vdots & \vdots & \ddots & \vdots \\ -s_n & s_n & \cdots & 0 \end{pmatrix}$$

Factors in this Jacobian matrix are expressed as follows.

$$s_i = \frac{u^i_{gg} - u^i_{xg}}{u^i_{xx} - u^i_{gx} + u^i_{gg} - u^i_{xg}} \qquad i = 1, 2, \cdots, n \tag{3}$$

A matrix S is also defined as follows.

$$S \equiv -R \tag{4}$$

On the other hand, total differentiation of the first order condition (1) brings us the following equations.

$$\left(1 + \frac{u^i_{gg} - u^i_{xg}}{u^i_{xx} - u^i_{gx}}\right) \frac{dg_i}{dw_i} = 1 \qquad i = 1, 2, \cdots, n$$

Because of the budget constraint of the player i and the *Assumption* 1, we have,

$$0 < \frac{dg_i}{dw_i} < 1 \qquad i = 1, 2, \cdots, n.$$

Those two equations mean,

$$v_i \equiv \frac{u^i_{gg} - u^i_{xg}}{u^i_{xx} - u^i_{gx}} > 0 \qquad i = 1, 2, \cdots, n \tag{5}$$

Bring together (3) and (5), we finally have,

$$0 < s_i = \frac{v_i}{1 + v_i} < 1 \qquad i = 1, 2, \cdots, n. \tag{6}$$

First, let us prove the following lemma.

Lemma 1 : For the matrix $I - S$, $|I - S| > 0$ is a necessary and sufficient condition of satisfying Hawkins-Simon condition.

Proof:
First, let us shows Hawkins-Simon condition (See, for example, Takayama [12]).

Let us define $S_k, k = 1, 2, \cdots, n$ as follows.

$$S_k = \begin{pmatrix} 0 & s_1 & s_1 & \cdots & s_1 \\ s_2 & 0 & s_2 & \cdots & s_2 \\ s_3 & s_3 & 0 & \cdots & s_3 \\ \vdots & \vdots & \ddots & \vdots & \\ s_k & s_k & s_k & \cdots & 0 \end{pmatrix}.$$

Let H_k be the kth successive principal minor of $I-S$ and I_k be a k dimension unit matrix. That is,

$$H_k = |I_k - S_k| \qquad k = 1, 2, \cdots, n$$

Using the above notation and $S \geq 0$, Hawkins-Simon condition is expressed as follows.

$$H_k > 0 \qquad k = 1, 2, \cdots, n$$

Then we can immediately have the necessary condition. That is, if Hawkins-Simon condition holds,

$$|I - S| = H_n > 0.$$

Nex let us prove the sufficiency. This is to show that if $|I - S| > 0$ holds, Hawkins-Simon condition also holds.

Since $0 < s_i < 1, i = 1, 2, \cdots, n$, we have,

$$H_1 = 1 > 0, H_2 = 1 - s_1 s_2 > 0.$$

For $k \geq 2$, H_k is generally expressed as follows.

$$H_k = \begin{vmatrix} 1 & -s_1 & -s_1 & \cdots & -s_1 \\ -s_2 & 1 & -s_2 & \cdots & -s_2 \\ -s_3 & -s_3 & 1 & \cdots & -s_3 \\ \vdots & \vdots & \ddots & \vdots & \\ -s_k & -s_k & -s_k & \cdots & 1 \end{vmatrix}.$$

First, by the manipulation for kth column, we can transform the above determinant as follows,

$$H_k = \begin{vmatrix} 1+s_1 & 0 & 0 & \cdots & -s_1 \\ 0 & 1+s_2 & 0 & \cdots & -s_2 \\ 0 & 0 & 1+s_3 & \cdots & -s_3 \\ \vdots & \vdots & \ddots & \vdots & \\ -s_k-1 & -s_k-1 & -s_k-1 & \cdots & 1 \end{vmatrix}.$$

Then, by the manipulation for kth collumn and each ith column, we have,

$$H_k = \begin{vmatrix} 1+s_1 & 0 & 0 & \cdots & 0 \\ 0 & 1+s_2 & 0 & \cdots & 0 \\ 0 & 0 & 1+s_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ -s_k-1 & -s_k-1 & -s_k-1 & \cdots & M_k \end{vmatrix},$$

where,

$$M_k = 1 - \left(\sum_{i=1}^{k-1} \frac{s_i}{1+s_i} \right) (1 + s_k) \qquad k = 2, 3, \cdots, n. \tag{7}$$

Thus we have H_k as follows.

$$H_k = \prod_{i=1}^{k-1}(1+s_i)M_k \qquad k = 2, 3, \cdots, n \tag{8}$$

For $k = 3, 4, \cdots, n$,

$$\begin{aligned} H_{k-1} - H_k &= \prod_{i=1}^{k-2}(1+s_i)M_{k-1} - \prod_{i=1}^{k-1}(1+s_i)M_k \\ &= \{M_{k-1} - (1+s_{k-1})M_k\}\prod_{i=1}^{k-2}(1+s_i) \\ &= s_k\left(\sum_{i=1}^{k-1}\frac{s_i}{1+s_i}\right)\prod_{i=1}^{k-1}(1+s_i). \end{aligned} \tag{9}$$

The right hand side of the above equation is strictly positive. Moreover $1 = H_1 > H_2$. Therefore,

$$H_{k-1} > H_k \qquad k = 2, 3, \cdots, n. \tag{10}$$

Let us suppose $|I - S| > 0$. This directly means $H_n > 0$. Then we have

$$H_1 > H_2 > \cdots > H_{n-1} > H_n > 0. \tag{11}$$

This shows Hawkins-Simon condition holds. Thus $|I - S| > 0$ is the sufficient condition for satisfying Hawkins-Simon condition. (Q.E.D.)

Then we have to prove following lemma.

Lemma 2 : Let $\lambda_i, i = 1, 2, \cdots, n$ be the eigenvalues values of R. Then the matrix $I - S$ satisfies Hawkins-Simon condition if and only if $|\lambda_i| < 1, i = 1, 2, \cdots, n$.

Proof:
Let all eigenvalues of S be $\omega_i, i = 1, 2, \cdots, n$. Suppose that $I - S$ satisfies Hawkins-Simon condition. This is equivalent to the fact that there is a Frobenius root ω_f such that $0 < \omega_f < 1$ and $|\omega_i| \leq \omega_f, i = 1, 2, \cdots, n$ (See, for example, Theorem 4.D.2 in Takayama [12]). On the other hand, $S = -R$ means

$$\{-\omega_i | i = 1, 2, \cdots, n\} = \{\lambda_i | i = 1, 2, \cdots, n\}.$$

Therefore, equivalently the matrix R has a negative simple root λ_f and

$$|\lambda_i| \leq |\lambda_f| < 1, i = 1, 2, \cdots, n.$$

A half of the lemma has proved.

Othe other hand, if $|\lambda_i| < 1, i = 1, 2, \cdots, n$, then as shown in the above equivalent transformation, $|\omega_i| < 1, i = 1, 2, \cdots, n$ is also satisfied. Since S is a nonnegative matrix, there is a Frobenius root $\omega_f(> 0)$ and $|\omega_i| \leq \omega_f < 1, i =$

$1, 2, \cdots, n$. This means equivalently that Hawkins-Simon condition holds. The other half of the lemma has proved. (Q.E.D.)

Now we can prove the stability theorem for the Nash equilibrium.

Theorem 1 : (Stability conditions of Nash equilibrium)
Under *Assumption* 1 and *Assumption* 2, the Nash equilibrium in the private provision of public goods is locally stable if and only if $|I - S| > 0$ is satisfied.

Proof:
Lemma 1 and *Lemma* 2 show that $|I - S| > 0$ is satisfied if and only if the absolute values of all eigenvalues of R is strictly less than 1. From Ostrowski theorem (See for example Ortega [10]), it means that the Nash equilibrium is locally stable. On the other hand, if the Nash equilibrium is locally stable, the absolute values of all eigenvalues are strictly less than 1 (See, for example, Li [7] or Sizdarovszky [11]). Thus, from *Lemma* 1 and *Lemma* 2, this is equivalent to $|I - S| > 0$. (Q.E.D.)

Let us add some remarks on this theorem. First, If $n = 2$, then $|I - S| = 1 - s_1 s_1 > 0$. Thus the Nash equilibrium is inevitably stable. This fact was originally shown by Theocharis [13]. If $n \geq 0$, we have both stability cases and instability cases. We show some numerical examples in Appendix.

Second, the adjustment processes of our model are strictly based on reaction functions. We do not introduce any other arbitrary adjustment processes so as to adjust a portion of the difference between the current status and the equilibrium.

Third, this stability theorem is equivalently applicable to the stability of Cournot-Nash equilibrium and to the stability of Nash equilibrium for using commons. For the problem of commons, we can easily confirm the stability conditions of Nash equilibrium for the model shown in Gibbons [5].

4 Public-spiritedness and Partial Stability

In the following sections, we show some natural extensions of the *Theorem* 1. First , let us show some definitions.

In our stability theorem, $s_i, i = 1, 2, \cdots, n$ play important roles. Since $s_i = dx_i/dw_i$ and $dx_i/dw_i + dg_i/dw_i = 1$ for all players, s_i is a standardized self-interestedness of a player i. It means that if a player i increases s_i, he or she increases the proportion of private goods for additional increase of income. On the other hand, $1 - s_i$ is seen as public-spiritedness of the player i. It means conversely that if a player i increases $1 - s_i$, he or she increases the proportion of public goods for additional increase of income. In other words, it is the marginal propensity to consume public goods. Let us write this definition explicitly.

Definition 1 : (Public-spiritedness)
$1 - s_i$ is public-spiritedness of individual i.

Next consider the principal minors $H_k, k = 1, 2, \cdots, n$ of $|I - S|$. Assume that the society consist of n individuals and each of them is differentiate by the number $1, 2, 3, \cdots, n$. In the society, without loss of generality, we suppose that there are groups which consist of k members $\{1, 2, \cdots, k\}$and we denote the public-spiritedness for each individuals as $1 - s_i, i = 1, 2, \cdots, n$. Then let us define the stability potential of group k.

Definition 2 : (Stability potential)
$H_k = |I_k - S_k|$ is the stability potential of group k. The group k consist of members $\{1, 2, \cdots, k\}$, where $1 \leq k \leq n$, and in the case of $k = n$, the group k is equivalent to the society.

Now we can prove the following theorem.

Theorem 2 : (Stability potential and public-spiritedness)
Under *Assumption* 1 and *Assumption* 2, if the public-spiritedness of a member of group k is increased, the stability potential of the group is also increased.

Proof:
Let us suppose $j \in \{1, 2, \cdots, k\}$. Since,

$$\frac{\partial H_k}{\partial s_j} = -\prod_{i=1, i \neq j}^{k} (1 + s_i) \left(\sum_{i=1, i \neq j}^{k} \frac{s_i}{1 + s_i} \right) < 0, \tag{12}$$

the increase of $1 - s_j$ causes the increase of H_k. (Q.E.D.)

Definition 3 : (Partial stability)
Suppose that there is a group k such that the other members who do not belong to the group k constantly provide public goods at the level of the Nash equilibrium, and the members belonging to the group k can exclusively and freely change the amount of provision of public goods based on their reaction functions. If the adjustment by the members of group k is locally stable, then the group k is partially stable in the society.

Then we can prove the following theorem.

Theorem 3 : (Maximum group in partially stable groups)
Under *Assumption* 1 and *Assumption* 2, if the Nash equilibrium in the provision of public goods for a total society is not locally stable, then there exists the maximum group K such that those group k for $2 \leq k \leq K$ are partially stable and those group k for $K < k \leq n$ are not partially stable.

Proof:
Under our assumptions, as shown in the proof of *Lemma* 1, the following inequality holds.

$$1 = H_1 > H_2 > \cdots > H_{n-1} > H_n. \tag{13}$$

Suppose $H_n < 0$, then there exists a K such that for $1 < k \leq K$,

$$1 = H_1 > H_2 > \cdots > H_K > 0,$$

and,

$$0 > H_{K+1} > H_{K+2} > \cdots > H_n.$$

For any $k \in \{1, 2, \cdots, K\}$, if individuals $\{k+1, k+2, \cdots, n\}$ constantly provide public goods given in the Nash equilibrium, then $H_k > 0$ means that the iteration system composed only by individuals $\{1, 2, \cdots, k\}$ locally stable because of *Theorem* 1. On the other hand, For any $k \in \{K+1, \cdots, n\}$, $H_k < 0$ means the iteration system composed only by individuals $\{1, 2, \cdots, k\}$ locally instable because of *Theorem* 1. Therefore K is the maximum group that satisfies the conditions in this theorem. (Q.E.D.)

The premise that the other members provide public goods at the level of the Nash equilibrium seems to be restrictive. Suppose that, for a group k, the members who do not belong to this group constantly supply certain levels of public goods freely. Although there should be a Nash equilibrium, the Nash equilibrium varies as the configuration of supply by the other members. Moreover the stability potentials varies too.

It is, however, not so tragic. If the public-spiritedness of all members is constant with the change of income level, then the behavior of the members who do not belong to the group does not affect the stability potential of the group. Then we can eliminate the assumption that the other members provide public goods at the level of Nash equilibrium.

The other issue to be considers is the case that a new member joins in the society. Suppose that the new member is denoted by $n+1$. Obviously we can conclude that for new stability potential $H_n > H_{n+1}$ holds. However for the old potential H'_n before joining $n+1$, and H_{n+1}, $H'_n > H_{n+1}$ dose not necessarily holds. This is because that the public goods provided by $n+1$ may change every $H_k, k = 1, 2, \cdots, n$. If we assume that for members $\{1, 2, \cdots, n\}$, $ds_i/dG_{i-1} \geq 0$, then the joining of a new member inevitably decreases every $H_k, k = 1, 2, \cdots, n$ compared with the old potential, and $H'_n > H_n > H_{n+1}$ holds. Therefore we easily compare the old situation of the society and the new society after joining a new member. For example, since the following inequalities holds,

$$H'_1 > H'_2 > \cdots > H'_n > H_{n+1},$$

the joining of new member decrease the stability potential of the society.

5 Concluding Remarks

We have proved that the unique Nash equilibrium in the private provision of public goods is not necessarily stable and that the stability crucially depends upon the stability potential that the society has. The normalized propensity to consume of public goods expresses the public-spiritedness for a member of the society. If a person has strong public-spiritedness, he or she is strongly interested in public goods and tries to pay much more for public goods. This strong public-spiritedness ensures high stability potential for any group that he or she belongs to.

We have defined the partial stability of the Nash equilibrium. Even if total society is not stable, we can find a smaller group that the adjustment by the members of the group is ensured to be locally stable under the condition that the other members in the society constantly provide public goods at the level of the Nash equilibrium. Once we have found a maximum group that is partially stable, any group that the members of the group are belong to the maximum partially stable group is also stable.

This fact means that the stability of the Nash equilibrium depends on the scale in terms of the number of the members who belong to a group or a society. Larger societies or groups, they have the tendency to be more instable. Therefore we have to regulate free adjustment of expense for public goods by the members. That is, we are required to be invariable for that expense. This is an important reason why a tax system is introduced to our liberalistic societies. Furthermore our governments have been appeared to sustain this tax system.

Appendix: Numerical Examples

Let us see some numerical examples. There are four members in a society. They are differentiated by subscripts $\{1,2,3,4\}$. The utility functions are Cobb-Douglas type as shown below.

$$\begin{aligned} u_1 &= (w_1 - g_1)^{\alpha}(g_1 + G_{-1})^{1-\alpha} \\ u_2 &= (w_2 - g_2)^{\beta}(g_2 + G_{-2})^{1-\beta} \\ u_3 &= (w_3 - g_3)^{\gamma}(g_3 + G_{-3})^{1-\gamma} \\ u_4 &= (w_4 - g_4)^{\delta}(g_4 + G_{-4})^{1-\delta} \end{aligned}$$

where $0 < \alpha, \beta, \gamma, \delta < 1$.Then we have the following reaction function.

$$\begin{pmatrix} g_1^{t+1} \\ g_2^{t+1} \\ g_3^{t+1} \\ g_4^{t+1} \end{pmatrix} = \begin{pmatrix} 0 & -\alpha & -\alpha & -\alpha \\ -\beta & 0 & -\beta & -\beta \\ -\gamma & -\gamma & 0 & -\gamma \\ -\delta & -\delta & -\delta & 0 \end{pmatrix} \begin{pmatrix} g_1^{t} \\ g_2^{t} \\ g_3^{t} \\ g_4^{t} \end{pmatrix} + \begin{pmatrix} (1-\alpha)w_1 \\ (1-\beta)w_2 \\ (1-\gamma)w_3 \\ (1-\delta)w_4. \end{pmatrix}$$

Since the Jacobian of this system is composed by scalar values, we can expect that the local stability simultaneously means global stability.

We have the stability potential of groups as follows.

$$H_2 = \begin{vmatrix} 1 & -\alpha \\ -\beta & 1 \end{vmatrix},\ H_3 = \begin{vmatrix} 1 & -\alpha & -\alpha \\ -\beta & 1 & -\beta \\ -\gamma & -\gamma & 1 \end{vmatrix},\ H_4 = \begin{vmatrix} 1 & -\alpha & -\alpha & -\alpha \\ -\beta & 1 & -\beta & -\beta \\ -\gamma & -\gamma & 1 & -\gamma \\ -\delta & -\delta & -\delta & 1 \end{vmatrix}$$

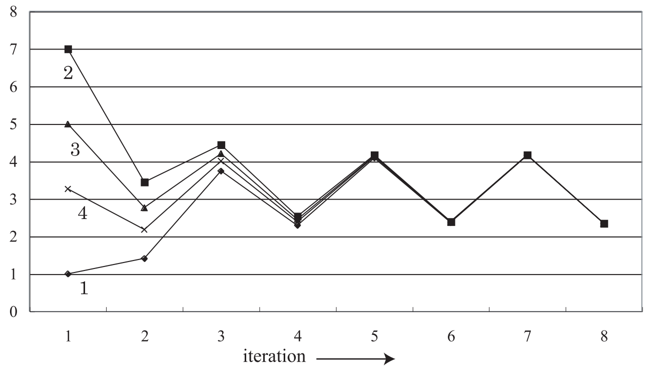

Figure 1: Adjustment of the total society

We simply assume that,

$$\alpha = \beta = \gamma = \delta = 0.34, \quad w_1 = w_2 = w_3 = w_4 = 10.$$

We can calculate the stability potential as follows.

$$H_2 = 0.8844, \quad H_3 = 0.574592, \quad H_4 = -0.048122$$

Therefore we see that the group 2 and the group 3 are partially stable. However the society is instable because of $H_4 < 0$. Let us confirm this theoretical inference. We have the Nash equilibrium as follows.

$$\begin{pmatrix} g_1^* = 0.32673 \\ g_2^* = 0.32673 \\ g_3^* = 0.32673 \\ g_4^* = 0.32673 \end{pmatrix}$$

Let us give the initial state as follows.

$$\begin{pmatrix} g_1^0 \\ g_2^0 \\ g_3^0 \\ g_4^0 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 5 \\ 0.32673 \end{pmatrix}$$

The quantity of public goods are spontaneously provided by the members in each period 0. Then members provide public goods based on their reaction function. The path of those iterations is shown in Figure 1. This figure show the iterations until the 8th period. Wherether stable or instable is not clear on the figure. If we see more periods after that period, we can see the instability of the system. Figure 2 show the path until 74th periods. The instability of this society is clearly shown in this figure. The reactions of players show strong syncronization.

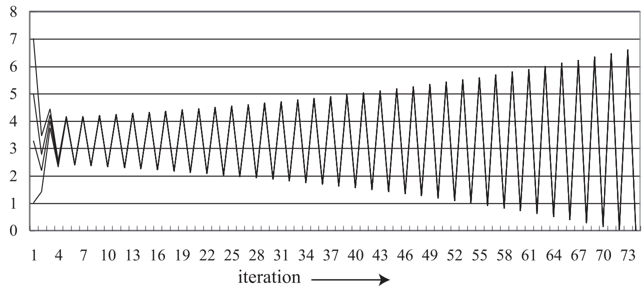

Figure 2: Instability of the Nash equilibrium for the society

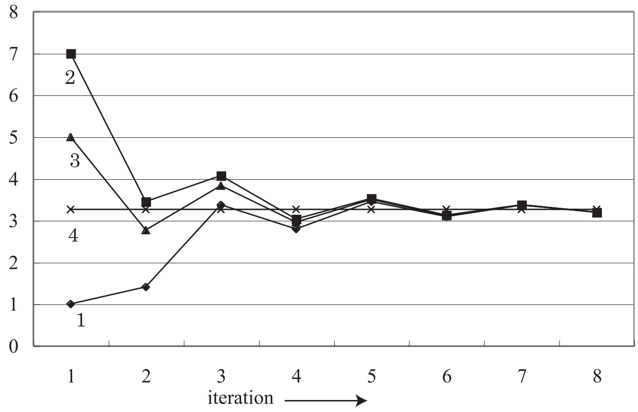

Figure 3: Partial stability of the society ($K = 3$)

Nex let us show the partial stability of this society. The level of the provision of public goods by member 4 is fixed at the level of the Nash equilibrium. The path of the iteration is shown in Figure 3.

We have assumed homogeneiety of for all members. They have the same parameters for utility functions. However, if one more member joins in the adjustment process, the system becomes instable.

Next let us examin the relationship between the size of a society and the stability potential of the society. Figure 4 show the value of $|I - S|$ varying with the scale of the society. In the figure, three ceses are shown. The first case is that the parameters of members in the society are expressed equivalently as

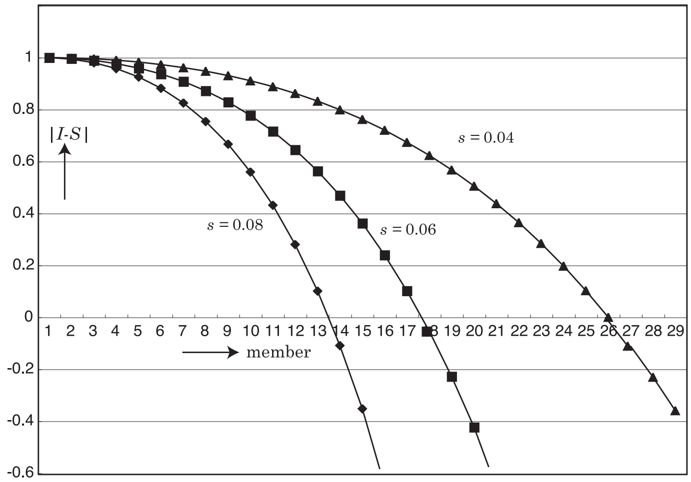

Figure 4: Scale of the society and stability potential

$s_i = 0.04, \quad i = 1, 2, \cdots, n$. That is,

$$H_n = \begin{vmatrix} 1 & -0.04 & -0.04 & \cdots & -0.04 \\ -0.04 & 1 & -0.04 & \cdots & -0.04 \\ -0.04 & -0.04 & 1 & \cdots & -0.04 \\ \vdots & \vdots & \ddots & \vdots & \\ -0.04 & -0.04 & -0.04 & \cdots & 1 \end{vmatrix}.$$

The muximum group of partially stable is 25, that is $K = 25$. If the parameters are equivalently $s_i = 0.06$, then $K = 17$. In case of $s_i = 0.08$, $K = 13$. Those facts are natural results expected by the theorems in this paper.

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Mathematical notes for referees

1. Transformation of equation (9)

$$
\begin{aligned}
H_{k-1} - H_k &= \prod_{i=1}^{k-2}(1+s_i)M_{k-1} - \prod_{i=1}^{k-1}(1+s_i)M_k \\
&= \{M_{k-1} - (1+s_{k-1})M_k\} \prod_{i=1}^{k-2}(1+s_i)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& M_{k-1} - (1+s_{k-1})M_k \\
= \;& 1 - \left(\sum^{k-2} \frac{s_i}{1+s_i}\right)(1+s_{k-1}) - 1 - s_{k-1} + (1+s_{k-1})\left(\sum^{k-1} \frac{s_i}{1+s_i}\right)(1+s_k) \\
= \;& -\left(\sum^{k-2} \frac{s_i}{1+s_i}\right)(1+s_{k-1}) + \left(\sum^{k-2} \frac{s_i}{1+s_i}\right)(1+s_{k-1})(1+s_k) \\
& -s_{k-1} + \frac{s_{k-1}}{1+s_{k-1}}(1+s_{k-1})(1+s_k) \\
= \;& s_k\left(\sum^{k-2} \frac{s_i}{1+s_i}\right)(1+s_{k-1}) - s_{k-1} + s_{k-1}(1+s_k) \\
= \;& s_k\left(\sum^{k-2} \frac{s_i}{1+s_i}\right)(1+s_{k-1}) + s_k s_{k-1} \\
= \;& s_k\left(\sum^{k-2} \frac{s_i}{1+s_i} + \frac{s_{k-1}}{1+s_{k-1}}\right)(1+s_{k-1}) \\
= \;& s_k\left(\sum^{k-1} \frac{s_i}{1+s_i}\right)(1+s_{k-1})
\end{aligned}
$$

Therefore,

$$
H_{k-1} - H_k = s_k\left(\sum_{i=1}^{k-1} \frac{s_i}{1+s_i}\right)\prod_{i=1}^{k-1}(1+s_i).
$$

2. Derivation of equation (12)

First, from (7) and (8), immediately we have,

$$
\frac{\partial H^k}{\partial s_k} = -\prod_{i=1}^{k-1}(1+s_i)\left(\sum_{i=1}^{k-1} \frac{s_i}{1+s_i}\right) < 0.
$$

Suppose $j \neq k$. Then,

$$
\begin{aligned}
&\frac{\partial H^k}{\partial s_j} \\
= \quad & \prod_{i \neq j}^{k-1}(1+s_i)M^k - \prod_{i=1}^{k-1}(1+s_i)\frac{1+s_k}{(1+s_j)^2} \\
= \quad & \prod_{i \neq j}^{k-1}(1+s_i) - \prod_{i \neq j}^{k-1}(1+s_i)\left(\sum_{i=1}^{k-1}\frac{s_i}{1+s_i}\right)(1+s_k) - \prod_{i=1}^{k-1}(1+s_i)\frac{1+s_k}{(1+s_j)^2} \\
= \quad & \prod_{i \neq j}^{k-1}(1+s_i)\left\{1 - \sum_{i=1}^{k-1}\frac{s_i}{1+s_i}(1+s_k) - \frac{1+s_k}{1+s_j}\right\} \\
= \quad & \prod_{i \neq j}^{k-1}(1+s_i)\left\{1 - \sum_{i \neq j}^{k-1}\frac{s_i}{1+s_i}(1+s_k) - \frac{s_j}{1+s_j}(1+s_k) - \frac{1+s_k}{1+s_j}\right\} \\
= \quad & \prod_{i \neq j}^{k}(1+s_i)\left\{\frac{1}{1+s_k} - \sum_{i \neq j}^{k-1}\frac{s_i}{1+s_i} - \frac{s_j}{1+s_j} - \frac{1}{1+s_j}\right\} \\
= \quad & \prod_{i \neq j}^{k}(1+s_i)\left\{-\sum_{i \neq j}^{k-1}\frac{s_i}{1+s_i} + \frac{1}{1+s_k} - 1\right\} \\
= \quad & \prod_{i \neq j}^{k}(1+s_i)\left\{-\sum_{i \neq j}^{k-1}\frac{s_i}{1+s_i} - \frac{s_k}{1+s_k}\right\} \\
= \quad & \prod_{i \neq j}^{k}(1+s_i)\left(-\sum_{i \neq j}^{k}\frac{s_i}{1+s_i}\right) \\
= \quad & -\prod_{i \neq j}^{k}(1+s_i)\left(\sum_{i \neq j}^{k}\frac{s_i}{1+s_i}\right).
\end{aligned}
$$

Thus we have derived (12).